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Boundary integral equation solution of viscous flows with free surfaces

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Summary

Solutions of the biharmonic equation governing steady two-dimensional viscous flow of an incompressible Newtonian fluid are obtained by employing a direct biharmonic boundary integral equation (BBIE) method in which Green's theorem is used to reformulate the differential equation as a pair of coupled integral equations which are applied only on the boundary of the solution domain.

An iterative modification of the classical BBIE is presented which is able to solve a large class of (nonlinear) viscous free surface flows for a wide range of surface tensions. The method requires a knowledge of the asymptotic behaviour of the free surface profile in the limiting case of infinite surface tension but this can usually be obtained from a perturbation analysis. Unlike space discretisation techniques such as finite difference or finite element, the BBIE evaluates only boundary information on each iteration. Once the solution is evaluated on the boundary the solution at interior points can easily be obtained.

1. Introduction

Free surfaces occur in a large range of physical phenomena – gravity waves, flow through porous media, forces on floating bodies, etc. In all such cases, the problem is complicated by the fact that the domain of solution has an unknown boundary on which known conditions are to be imposed. For potential problems, these are the kinematic and dynamic conditions whereas for viscous flows there is, in addition to the kinematic condition, the somewhat more complex requirement of continuity of the stress tensor across the free surface.

Many authors have successfully tackled such problems using a variety of numerical techniques, for example the finite difference (FD) method, the finite element (FE) method, boundary fitted coordinates, Green's functions etc and an excellent review of these methods is given by Yeung [1].

This paper deals with the classical boundary integral equation (BIE) method which has the advantage over space discretisation techniques of reducing the dimension of the problem by one, the equations having been integrated once analytically by application of Green's theorem. This, together with a knowledge of the fundamental solutions of Laplace's equation (and the biharmonic equation, when the flow is viscous) enables us to reformulate the original differential equation and boundary conditions as a coupled system of integral equations applied on the boundary only. The effect of this reformulation is to greatly reduce requirements in both computer storage and code [2]. Thus the BIE has the advantage over the FD and FE methods in that information regarding flow variables is evaluated only on the boundary and not at a number of interior points.

Successful BIE solutions of potential (Laplacian) free surface flows were obtained by Liggett [3], Niwa et al. [4] and Longuet-Higgins and Cokelet [5]. The first two authors solve for the profile of the seepage of groundwater through a porous dam. The third author employs a time-dependent Lagrangian BIE formulation for following the formation and evolution of breaking waves. In these works the free surface location is obtained by an iterative process based on the satisfaction of the kinematic condition which provides an explicit relationship between known BIE variables. For viscous flows however, the available conditions provide no such explicit relationship.

In the present work a direct biharmonic BIE (BBIE) method is presented in which the viscous flow variables are stream function, velocity, vorticity and vorticity gradient. We shall later see that using this formulation we may obtain explicit forms for the kinematic and shear stress conditions in terms of the BBIE variables. However, the normal stress condition includes the velocity variable in terms of a second spatial derivative and so care must be taken when evaluating this derivative numerically.

The most recent developments in the field of viscous free surface flows appear to have been achieved using variations and modifications of the FE method, see for example [6–9]. The work of Silliman [6] pioneered problems in slot coating flow and this was extended by Saito and Scriven [8] to accommodate film flows with highly bent menisci by combining polar and Cartesian parametrisations of the meniscus shape. By performing an integration by parts of the equations of motion across the free boundary Ruschak [7] was able to eliminate the explicit appearance of surface curvature in the equations, thus allowing a piecewise linear approximation of the surface profile even when surface tension effects are dominant. Frederiksen and Watts [9] presented a more sophisticated approach which was able to deal successfully with time dependent viscous flows with free surfaces. Their technique was essentially an implicit time-stepping method and so was stable even for relatively large time steps.

In problems where the location of the free surface is unknown, this fact is "compensated" for by the knowledge of *three* boundary conditions on the free surface: on fixed boundaries one is only ever supplied with two such conditions. However, one may apply only two of the conditions directly into any particular numerical scheme and use the third as the criterion for the location of the free surface. All of the papers [6–9] apply the kinematic condition as an essential free surface condition but vary in their choice of second condition: some choose that on shear stress, others that on normal stress. In the absence of any detailed analysis of the effects of this choice we must treat the question of which second condition to apply as one which remains unresolved. In the present work we enforce the kinematic and shear stress conditions and iterate on the normal stress condition to obtain the free surface location.

The iterative scheme applied in this paper requires a knowledge of the form of the free surface in the limiting case of large surface tension, T say. This may be obtained in the manner described by Richardson [10] by expressing the flow variables and surface profile as expansions of the parameter $C = 1/T$ so that $C \ll 1$. Insertion of these expansions into the known boundary conditions yields the asymptotic form of the free surface for large T and it is this form which is used to initiate the iterative procedure, whatever the value of T. Thus we would expect the convergence to be faster for the larger values of T. The method is therefore suited to all such problems for which this asymptotic behaviour is obtainable.

The algebraic system of equations generated by the BBIE is nonlinear not in the classical sense – i.e. in that the flow variables occur as explicit nonlinear terms – but rather in that the BBIE matrix of coefficients of the flow variables are themselves dependent upon the location of the free surface and so are also unknown. Thus standard iterative techniques for solving nonlinear equations will not suffice. An algorithm will be presented which converges on the correct boundary location as well as solving for the unknown flow variables.

In order to test the application of this BBIE method to viscous free boundary problems we consider the problem of a two-dimensional jet expanding from between two semi infinite parallel plates. A detailed analytical study of this problem has been made by Richardson [10] and there are FE calculations due to Ruschak [7] which we will use for comparison. The results from the present work and those from [7] are seen to be in good agreement.

2. Formulation

Governing equations

For steady state two-dimensional creeping flow of an incompressible Newtonian fluid we obtain, from the Navier-Stokes equations (neglecting inertia terms):

$$\nabla P = \nabla^2 \boldsymbol{u}, \tag{1}$$

$$\nabla \cdot \boldsymbol{u} = 0, \tag{2}$$

where P and $\boldsymbol{u}$ are the nondimensional fluid pressure and velocity respectively. Introducing the stream function ψ, the x and y components of velocity are then given by

$$u = \psi_y, \qquad v = -\psi_x, \tag{3}$$

respectively. From Eqns. (1), (2) and (3) it may be shown [11] that ψ satisfies the biharmonic equation

$$\nabla^4 \psi = 0. \tag{4}$$

Introducing the vorticity ω we rewrite Eqn. (4) in its coupled form

$$\nabla^2 \psi = \omega, \tag{5}$$

$$\nabla^2 \omega = 0. \tag{6}$$

To solve Eqns. (5) and (6) in the region Ω enclosed by boundary $\partial\Omega$ we first transform them into their equivalent integral representations. Denote the field point at which the solution is required by $p(x_0, y_0)$ and the general position on the boundary by $q(x, y)$ so that $p \in \Omega + \partial\Omega$ and $q \in \partial\Omega$. Define G_1 and G_2 to be the fundamental solutions of Laplace's equation and the biharmonic equation respectively. Then

$$\nabla^2 G_1(p, q) = \delta(|p - q|), \tag{7}$$

$$\nabla^4 G_2(p, q) = \delta(|p - q|) \tag{8}$$

where

$$|p-q| = \left[(x-x_0)^2 + (y-y_0)^2\right]^{1/2} \tag{9}$$

and δ is the Dirac delta function. The required fundamental solutions are then

$$G_1(p,q) = \frac{1}{2\pi}\log|p-q|, \tag{10}$$

$$G_2(p,q) = \frac{1}{8\pi}|p-q|^2[\log|p-q|-1]. \tag{11}$$

Applying a biharmonic form of Green's second identity [12] to the functions ψ and G_2 and employing the singular behaviour of G_2 as $p \to q$ gives

$$\begin{aligned}\xi(p)\psi(p) = \int_{\partial\Omega} \{ &\psi(q)G_{1n}(p,q) - \psi_n(q)G_1(p,q) \\ &+ \omega(q)G_{2n}(p,q) - \omega_n(q)G_2(p,q)\}\,\mathrm{d}q\end{aligned} \tag{12}$$

where Eqns. (5) and (6) have been used and the subscript n refers to differentiation with respect to the outward normal to Ω, $p \in \Omega + \partial\Omega$, $q \in \partial\Omega$ and $\mathrm{d}q$ denotes the differential increment of $\partial\Omega$ at q. The function $\xi(p)$ is obtained by a study of the behaviour of the fundamental solutions (10) and (11) as $p \to q$ and is given by

$$\xi(p) = \begin{cases} 0 & \text{if } p \notin \Omega + \partial\Omega \\ \dfrac{\alpha}{2\pi} & \text{if } p \in \partial\Omega \\ 1 & \text{if } p \in \Omega \end{cases} \tag{13}$$

where α is the angle included between the tangents to $\partial\Omega$ on either side of p. Now applying Green's second identity [12] to the functions ω and G_1 and employing the singular nature of G_1 as $p \to q$ gives

$$\xi(p)\omega(p) = \int_{\partial\Omega}\{\omega(q)G_{1n}(p,q) - \omega_n(q)G_1(p,q)\}\,\mathrm{d}q \tag{14}$$

where now Eqn. (6) has been used.

In the ensuing analysis, the subscript t denotes differentiation with respect to the unit tangent to $\partial\Omega$ and the subscript s with respect to the coordinate measured anticlockwise along $\partial\Omega$ from some fixed point on $\partial\Omega$. Here the (t, n) axes form a right-handed set so that, in an obvious notation,

$$\nabla^2_{(t,n)} f = \nabla^2_{(x,y)} f \tag{15}$$

for any sufficiently differentiable function f.

Equations (12) and (14) form the basis of the BBIE method. Solution of these equations depends upon the knowledge of a set of boundary conditions in terms of the BBIE variables of stream function ψ, boundary velocity ψ_n, vorticity ω and normal vorticity gradient ω_n.

Boundary conditions

To illustrate the BBIE formulation we shall consider the following specific problem, due originally to Richardson [10].

Incompressible Newtonian fluid flows from left to right in the semi-infinite channel bounded by the no-slip planes $-\infty < x \leqslant 0$, $y = \pm 1$. As $x \to -\infty$ the flow tends to a Poiseuille velocity profile. For $0 < x < +\infty$ the flow is bounded, in the absence of gravity, by the free surfaces $y = \pm\eta(x)$ i.e. the flow is symmetric about $y = 0$. As $x \to +\infty$ the velocity profile is constant throughout the region bounded by the upper and lower free surfaces. By employing a symmetry argument about the channel centreline $y = 0$ we need solve the problem only in the upper half channel $y \geqslant 0$, noting that the channel centreline is a boundary supporting zero tangential shear. The kinematic condition on the upper free surface $y = \eta(x)$ merely suggests that the free surface is a streamline – since then no fluid particles may cross the free surface. Suppose U and V are respectively the tangential and normal velocity components on the free surface $y = \eta(x)$. Then the shear stress condition at the free surface (assuming the region $|y| > |\eta(x)|$ to have negligible viscosity) gives

$$\sigma_{tn} = \frac{\partial U}{\partial n} + \frac{\partial V}{\partial t} = 0 \tag{16}$$

which, after inserting the relations in (3) gives

$$\psi_{nn} - \psi_{tt} = 0. \tag{17}$$

Furthermore, from Eqns. (6), (15) and (17) we obtain

$$\omega = 2\psi_{tt}. \tag{18}$$

Coyne and Elrod [13] show that

$$\left.\frac{\partial}{\partial t}\right|_{\partial\Omega} = \frac{\mathrm{d}}{\mathrm{d}s} = \cos\beta\frac{\mathrm{d}}{\mathrm{d}x}, \tag{19}$$

$$\left.\frac{\partial^2}{\partial t^2}\right|_{\partial\Omega} = \frac{\mathrm{d}^2}{\mathrm{d}s^2} - \kappa\left.\frac{\partial}{\partial n}\right|_{\partial\Omega}, \tag{20}$$

where κ is the local curvature of the free surface and β is the anti-clockwise angle between the tangent to the free surface and the x axis. Hence we have

$$\kappa = -\eta'' \cos^3\beta, \tag{21}$$

$$\tan\beta = \eta' \tag{22}$$

where a prime denotes differentiation with respect to x. From Eqns. (18), (20) and the kinematic boundary condition (essentially $\psi_s = \psi_{ss} = 0$) we obtain

$$\omega = -2\kappa\psi_n. \tag{23}$$

Hence Eqn. (23) provides a linear relationship between the boundary vorticity and velocity and is therefore the required form of the shear stress condition in terms of BBIE variables.

The kinematic and shear stress conditions may thus be obtained explicitly in terms of the BBIE variables and so may readily be enforced in Eqns. (12) and (14). The remaining condition on normal stress is used as the iterative criterion. We require the normal stress to be balanced by surface tension and fluid pressure so that

$$-T\kappa = p_a + \sigma_{nn} \tag{24}$$

where T is the nondimensional surface tension, p_a is the ambient pressure and σ_{nn} is the normal stress component. Differentiating Eqn. (24) with respect to t, substituting for σ_{nn} and employing the relations in (3) gives

$$T\kappa_t = p_t + 2\psi_{ntt} \tag{25}$$

whence, by virtue of Eqns. (1), (20) and (23)

$$T\kappa_t = \omega_n - \kappa\omega + 2\psi_{nss}, \tag{26}$$

This expression involves the vorticity, vorticity gradient and the second derivative of the boundary velocity with respect to s. Note that condition (26) contains ψ_{nss} in preference to ψ_{ntt} since the former may be numerically evaluated in terms of the BBIE values of ψ_n available on the free surface. The required boundary conditions are therefore

$$\psi = 1, \qquad \psi_y = 0 \quad \text{on } y = 1, \qquad x \leqslant 0, \tag{27a}$$

$$\psi = 0, \qquad \omega = 0 \quad \text{on } y = 0, \tag{27b}$$

$$\psi_x \to 0, \qquad \psi_y \to \tfrac{3}{2}(1 - y^2) \quad \text{as } x \to -\infty, \tag{27c}$$

$$\psi_x \to 0, \qquad \psi_y \to 1/\eta_\infty \quad \text{as } x \to +\infty, \tag{27d}$$

$$\left.\begin{aligned} &\psi = 1, \qquad \omega = -2\kappa\psi_n \\ &T\kappa_t = \omega_n - \kappa\omega + 2\psi_{nss} \end{aligned}\right\} \quad \text{on } y = \eta(x), \qquad x > 0, \tag{27e}$$

so that the full problem specification is as shown in Fig. 1. Conditions (27a) give the

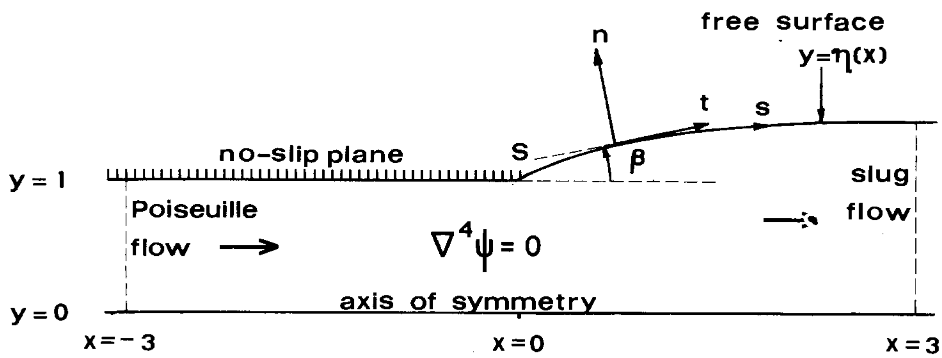

Figure 1. Problem geometry and boundary conditions.

steady stream potential and no-slip velocity on the upper plane, whilst conditions (27b) are those due to symmetry about the channel centreline. The enforced upstream Poiseuille flow is given in (27c); the downstream slug flow in (27d). The constant η_∞ in (27d) is the semi-jet width far downstream. Conditions (27e) are those applied on the free surface. Thus at each point on $\partial\Omega$ at least two boundary conditions in terms of BBIE variables are known.

Numerical solution

Applying Eqns. (12) and (14) at the points $\bar{q} \in \partial\Omega$ and $q \in \partial\Omega$ provides the following coupled nonlinear (because the boundary $\partial\Omega$ is as yet unknown) integral equations

$$\int_{\partial\Omega} \{ \psi(q) G_{1n}(\bar{q}, q) - \psi_n(q) G_1(\bar{q}, q) + \omega(q) G_{2n}(\bar{q}, q) - \omega_n(q) G_2(\bar{q}, q) \} \mathrm{d}q$$

$$- \xi(\bar{q}) \psi(\bar{q}) = 0, \qquad q, \bar{q} \in \partial\Omega, \tag{28}$$

$$\int_{\partial\Omega} \{ \omega(q) G_{1n}(\bar{q}, q) - \omega_n(q) G_1(\bar{q}, q) \} \mathrm{d}q - \xi(\bar{q}) \omega(\bar{q}) = 0, \qquad q, \bar{q} \in \partial\Omega. \tag{29}$$

Solution of the coupled Eqns. (28) and (29) subject to the boundary conditions (27) then provides a complete set of boundary information at each point $q \in \partial\Omega$. At this point note that conditions (27d) and (27e) are necessarily applied at finite distance, X say, both upstream and downstream of the separation point where the free surface leaves the no-slip plane, hereafter referred to as S. These conditions were applied at positions $x = \pm X$, the value of X being varied until a settled solution was obtained. It was found that taking $X > 3$ caused negligible change in the results presented. In fact, for this problem, virtually all flow characteristics change most rapidly in the region $-1 \leqslant x \leqslant 2$ (see Richardson [10]).

The complete set of boundary information thus obtained is inserted into Eqns. (12) and (14) to provide ψ and ω at the general field point $p \in \Omega + \partial\Omega$. In practice Eqns. (28) and (29) may rarely be solved analytically. We therefore adopt a numerical solution technique analogous to that employed by Symm [14] for solving Laplacian boundary value problems. The following description is that of the classical BBIE. Modification to include the effects of the free surface will be introduced later.

The boundary $\partial\Omega$ is subdivided into N straight line segments $\partial\Omega_j, j = 1, \ldots, N$ on which the functions ψ, ψ_n, ω and ω_n take the piecewise-constant values ψ_j, ψ_{nj}, ω_j and ω_{nj}. A discretised form of Eqns. (28) and (29) is then applied at the midpoint $q \equiv q_i$, $i = 1, \ldots, N$ of each interval. This generates a set of $2N$ simultaneous equations in the $2N$ unknown values of ψ_j, ψ_{nj}, ω_j and ω_{nj}. Solution of these algebraic equations then determines the remaining $2N$ boundary conditions which supplement the original boundary conditions (27). Applying the discretised forms of Eqns. (12) and (14) at the general field point $p \in \Omega + \partial\Omega$ determines $\psi(p)$ and $\omega(p)$ at any point in the solution domain.

Discretisation of the integral equations (28) and (29) means that, for example, the first term in the integrand in Eqn. (28) is approximated according to the following rule:

$$\int_{\partial\Omega} \psi(q) G_{1n}(\bar{q}, q) \mathrm{d}q \to \sum_{j=1}^{N} \psi_j \int_{\partial\Omega_j} G_{1n}(\bar{q}, q) \mathrm{d}q. \tag{30}$$

All of the integrals in the discretised equations were evaluated analytically to maintain accuracy at the present stage: the integrals associated with the G_2 and G_{2n} functions have previously been obtained numerically [15]. Details of the analytic integrations are given by Kelmanson [16].

It now remains to obtain the position of the free surface before using the above scheme to solve for the general flow.

Boundary location algorithm

As the surface tension tends to infinity, the free surface becomes a shear-free straight line extension of the no-slip plane $y = 1$, $x < 0$. A study of the outward normal force on this shear-free surface reveals that this force is everywhere positive, suggesting that a two dimensional jet emerging from between parallel no-slip planes into an inviscid atmosphere at zero Reynolds number would expand. Since in the absence of gravity there is no mechanism to counteract this expansion, we expect the surface profile $y = \eta(x)$, $x > 0$ to satisfy

$$\eta(0) = 1, \tag{31a}$$

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\eta(x) \to 0 \text{ as } x \to \infty, \qquad \text{all } n \geqslant 1. \tag{31b}$$

Defining the *swell* of the jet, α, as the increase in semi-channel width far downstream of S we have

$$\eta_\infty = 1 + \alpha \tag{32}$$

where η_∞ was introduced in condition (27d). It is convenient to introduce a functional form for $\eta(x)$ since then the value of κ_t in the normal stress boundary condition (26) may be evaluated analytically (via Eqns. (19) and (21)), thus preserving accuracy. We proceed to try to obtain such a form.

Richardson's [10] solution for large surface tension was obtained by expressing the free surface location as the linear expansion

$$\eta(x) = 1 + C\eta_1(x) + O(C) \tag{33}$$

where $C = 1/T$ so that $C \ll 1$. The substitution of Eqn. (33) into the stress conditions on the free surface gives $\eta'(0) > 0$ so that the free surface has a finite gradient at the separation point S. However, the analysis of Michael [17] shows that $\eta'(0)$ is necessarily zero if the normal stress on the free surface is to remain bounded at S. Moffatt [18] arrives at similar conclusions from the study of a flat plate being drawn into the free surface of a viscous fluid. This leads Richardson [10] to conclude that the perturbation in this region is necessarily singular, and that a more detailed understanding of the separation process is required before a satisfactory study of this region can be completed.

In the light of this uncertainty, we are unable to take account of the true nature of the solution in this region and so we must expect our numerical results to be in error near S. Since the main aim of this paper is to demonstrate how the BBIE method may be extended to solve free surface viscous flow problems, we shall employ the formal expansion of Eqn. (33) as a *first approximation* in the iterative scheme. Substitution of this expansion into the normal and shear stress boundary conditions provides the form of η_1, which requires numerical evaluation. We find that $\eta_1(x)$ in this case can be approximated by

$$\eta_1(x) \simeq 0.356 \tanh[x\epsilon(x)] \tag{34}$$

where $\epsilon(x)$ is a monotonic decreasing function of x with $\epsilon(0) \simeq 2.13$ and $\epsilon(x) \to 1.32$ as $x \to \infty$. Inserting into Eqn. (34) the function

$$\epsilon(x) = \epsilon_\infty + (\epsilon_0 - \epsilon_\infty)\exp(-\gamma x) \tag{35}$$

with $\epsilon_0 = 2.13$, $\epsilon_\infty = 1.32$ and $\gamma = 1.02$, gives close agreement with the results in [10].

The results of Eqns. (33), (34) and (35) suggest that we should approximate our free surface by functions of the form

$$\eta(x) = 1 + \alpha \tanh[x\epsilon(x)]. \tag{36}$$

Then choosing $\alpha = 0.356$ and $\epsilon = \epsilon(x)$ as suggested by Eqn. (35) should provide us with a first approximation to the free surface provided we are not in the neighbourhood of S. Note that the exact values of ϵ_0, ϵ_∞ and γ in Eqn. (35) are not important at this stage but serve to provide initial values for the ensuing iterative algorithm.

Now the geometry of the free surface describes by Eqn. (36) is such that $\eta'(0) \neq 0$ and therefore it does not satisfy the physical requirement of Michael [17]. We must assume that the behaviour of Eqn. (36) applies to the free surface only at distances greater than some small parameter, δ say, away from S. In $0 < x < \delta$, $\eta'(x)$ changes rapidly from $\eta'(0) = 0$ to $\eta'(\delta)$ finite. The unresolved question is then as to the magnitude of δ. Our approximation here is to assume that δ is infinitesimally small and that the free surface has infinite curvature at the point of separation. In this work no asympototic behaviour on the solution near S has been enforced, and the above approximation is seen to be a sufficent starting point for the ensuing iterative algorithm.

Step 1. Specify a required nondimensional surface tension T.

Step 2. Letting Γ represent the iteration number, specify M distinct free surfaces of the form suggested by Eqns. (34) and (35), namely

$$\eta_{m\Gamma}(x) = 1 + \alpha_m \tanh[x\epsilon_{m\Gamma}(x)], \qquad m = 1,\dots,M \tag{37}$$

where α_m is the swell of the surface $\eta_{m\Gamma}$ on the iteration Γ. In order to encourage convergence, the values of α_m were chosen in the range $0 < \alpha_m \leqslant 0.20$, the lower bound being a physical necessity and the upper being strongly suggested by the FE results of Ruschak [7] and Patten and Finlayson [19]. In the absence of any FE results for comparative purposes we would merely have expanded the range: in this case the FE results were used to aid the iterative process. In Eqn. (37) we also have

$$\epsilon_{m\Gamma}(x) = \epsilon_{m\Gamma}^{\infty} + \left(\epsilon_{m\Gamma}^{0} - \epsilon_{m\Gamma}^{\infty}\right)\exp(-\gamma_{m\Gamma} x). \tag{38}$$

If $\Gamma = 1$, then regardless of T we allow ϵ_{m1}^0, ϵ_{m1}^∞ and γ_{m1} to vary in the range of the parameters suggested by the analytic solution for large T. This range of variation is initially taken to be reasonably large so as not to enforce too rigid a form on the free surface profile. In fact, for $\Gamma = 1$, the ranges $1.4 \leqslant \epsilon_{m1}^{0} \leqslant 2.0$, $0.7 \leqslant \epsilon_{m1}^{\infty} \leqslant 1.3$ and $1.0 \leqslant \gamma_{m1} \leqslant 1.7$ were specified, for each T.

Step 3. With $\eta_{m\Gamma}$ specified, solve the problem via the classical BBIE enforcing the first two conditions in (27e) on the free surface. This generates the (incorrect) values of ψ_n, ω and ω_n on the free surface. It is found that the discrete values of ψ_n over the section of the free surface not in the neighbourhood of S admit a fitted curve of the form

$$\Psi_n(x) = ax^b \tag{39}$$

where $a = a(m, \Gamma, T)$ and $b = b(m, \Gamma, T)$ are characteristic of the current free surface, and $a, b > 0$. The continuous function ψ_n in Eqn. (39) is fitted to the discrete values of ψ_n by taking the logarithm of Eqn. (39) and using standard least-squares curve fitting procedures, details of which may be found in any elementary text on orthonormal functions. The approximation ψ_n was found to satisfy

$$\left|1 - \frac{\Psi_n}{\psi_n}\right| < 10^{-4}, \qquad \text{all } \Gamma \tag{40}$$

at each surface node at which Eqn. (43) was applied, showing the smoothness of the discrete values of ψ_n. We may now obtain accurate values of ψ_{nss} for insertion into the normal stress boundary condition (26). To achieve these, we use Eqns. (19) to obtain

$$\psi_{nss} \simeq \Psi_{nss} = \cos\beta \frac{\mathrm{d}}{\mathrm{d}x}\left[\cos\beta \frac{\mathrm{d}}{\mathrm{d}x}\Psi_n\right]. \tag{41}$$

Then Eqns. (39) and (41) provide us with the required values of ψ_{nss}.

Step 4. At each of K "test nodes" (x_k, y_k) on the free surface $\eta_{m\Gamma}$ evaluate the *residue*

$$R_{km\Gamma} = \omega_n - \kappa\omega + 2\psi_{nss} - T\kappa_t, \qquad k = 1,\dots,K, \tag{42}$$

which effectively measures the extent to which the normal stress condition is satisfied, $R_{km\Gamma}$ vanishing when the correct surface is obtained.

Step 5. Obtain the real constants $\beta_{m\Gamma}$, $m = 1,\dots,M$ which enable a vanishing linear combination of the residues to be obtained at each test node (x_k, y_k), $k = 1,\dots,K$. That is, find the $\beta_{m\Gamma}$ which satisfy

$$\left.\begin{aligned} &\sum_{m=1}^{M} R_{km\Gamma}\beta_{m\Gamma} = 0, \quad k = 1,\dots,K \qquad (43)\\ &\sum_{m=1}^{M} \beta_{m\Gamma} = 1 \qquad (44)\end{aligned}\right\} \text{all } \Gamma.$$

Equations (43) and (44) may be solved simultaneously for unique values of $\beta_{m\Gamma}$ provided we adhere to the additional constraint $K = M - 1$. Equation (44) is a scaling equation which provides unique values of $\beta_{m\Gamma}$ for Step 6.

Step 6. Take $\bar{\eta}_\Gamma$ as the new approximation to the free surface where

$$\bar{\eta}_\Gamma = \sum_{m=1}^{M} \beta_{m\Gamma}\eta_{m\Gamma}, \qquad \text{all } \Gamma \tag{45}$$

so that were the problem linear, $\bar{\eta}_\Gamma$ would be the correct free surface. Note that Eqn. (45) shows that Eqn. (44) is in fact one of mass conservation.

Step 7. In exactly the same manner as was used to obtain ϵ_0, ϵ_∞ and γ from the analytic solution, we fit a curve of the form

$$\bar{\eta}_\Gamma(x) = 1 + \bar{\alpha}_\Gamma \tanh[x\bar{\epsilon}_\Gamma(x)] \tag{46}$$

to the surface just obtained. Here $\bar{\alpha}_\Gamma$ is the swell of the "modified" surface, and

$$\bar{\epsilon}_\Gamma(x) = \bar{\epsilon}_\Gamma^\infty + (\bar{\epsilon}_\Gamma^0 - \bar{\epsilon}_\Gamma^\infty)\exp(-\bar{\gamma}_\Gamma x) \tag{47}$$

where the values of $\bar{\epsilon}_\Gamma^0$, $\bar{\epsilon}_\Gamma^\infty$ and $\bar{\gamma}_\Gamma$ are obtained by the aforementioned curve-fitting procedure.

Step 8. Evaluate the residues $\bar{R}_{k\Gamma}$, say, at the nodes (x_k, y_k), $k = 1, \ldots, K$ on the modified surface $\bar{\eta}_\Gamma$.

Step 9. The iteration is considered to have converged when *both*

$$(a)\left|1 - \frac{\bar{\alpha}_{\Gamma-1}}{\bar{\alpha}_\Gamma}\right| < 10^{-4}, \tag{48}$$

$$(b)|\bar{R}_{k\Gamma}| < 2 \times 10^{-4}, \qquad k = 1, \ldots, K. \tag{49}$$

If the criteria (48) and (49) are not satisfied, proceed to Step 10. If they are, the iteration is complete.

Step 10. Pass on to the next iteration number, $\Delta = \Gamma + 1$, and update the values of the parameters in Eqn. (38) by varying $\epsilon_{m\Delta}^0$ in the neighbourhood of $\bar{\epsilon}_\Gamma^0$, $\epsilon_{m\Delta}^\infty$ in the neighbourhood of $\bar{\epsilon}_\Gamma^\infty$, and $\gamma_{m\Delta}$ in the neighbourhood of $\bar{\gamma}_\Gamma$ for each $m = 1, \ldots, M$. (These variations are restricted to ever-decreasing ranges as the iteration proceeds.) Return to Step 2.

* * *

As in many iterative schemes, convergence is not necessarily guaranteed. However, the algorithm was tested over the large range of surface tensions 10^{-3}–10^{+3}: essentially from negligible to infinite T. For each T, the convergence of the numerical scheme was checked by employing discretisations comprising $N = 70$, 140 and 280 nodes. Choosing $M = 7$ and $K = 6$ was seen to be sufficient to obtain consistent solutions for the above parameter ranges. Obviously, M (and therefore K) would have to be increased for free surfaces on which the formation of waves was expected.

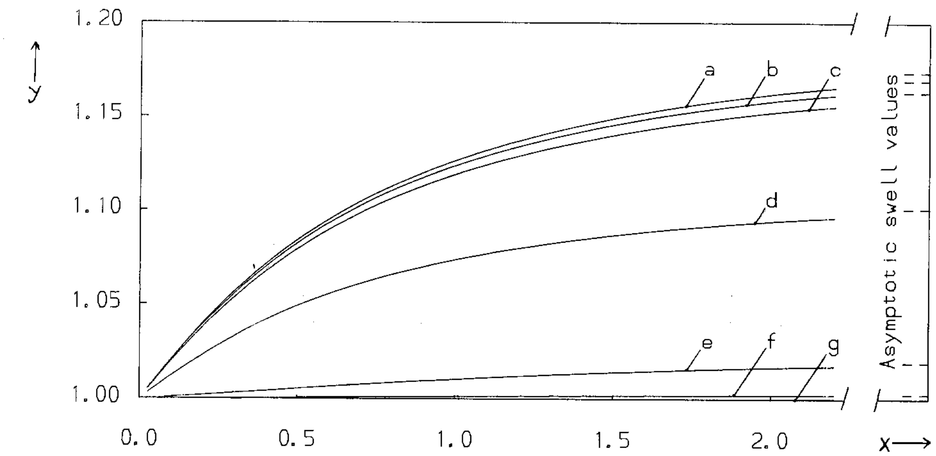

Figure 2. Free surface profiles for varying surface tensions T: (a) 0.001, (b) 0.01, (c) 0.1, (d) 1.0, (e) 10.0, (f) 100.0, (g) 1000.0.

3. Results and discussion

The iterative scheme outlined above was applied successfully to the problem of the two-dimensional expanding jet. The method was found to be convergent for the whole range of surface tensions considered, the convergence being faster for the finer discretisations. As expected, convergence was far more rapid for the larger surface tensions since the free surface profiles were then only small perturbations from the straight line extension of the solid plate. Converged solutions were obtained in an average of three to four iterations, the exceptional case of very small T and small N requiring as many as seven.

The free surface profiles generated by the 280 node BBIE are displayed graphically in Fig. 2 and are in agreement with those presented in [7], although as expected, neither the FE or the BBIE method shed any light on the solution behaviour near S.

In Fig. 3 we present the velocity distributions on the channel centreline $y = 0$ and the

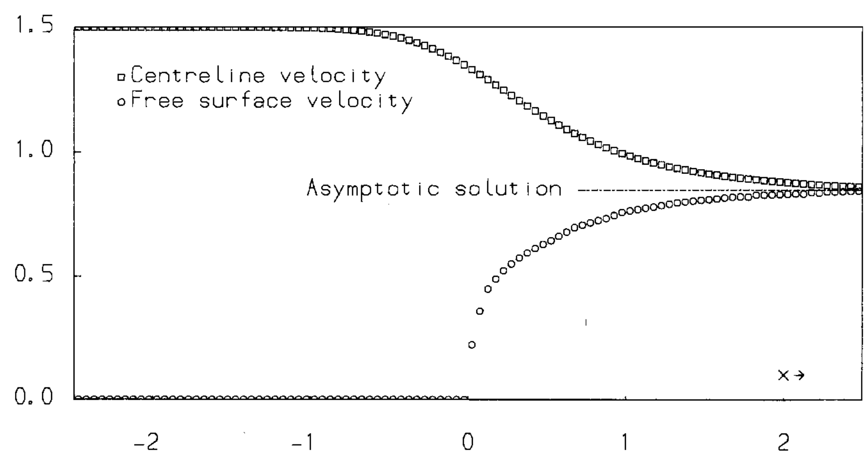

Figure 3. Velocity distributions on the channel centreline and free surface for $T = 0.001$ and $N = 280$.

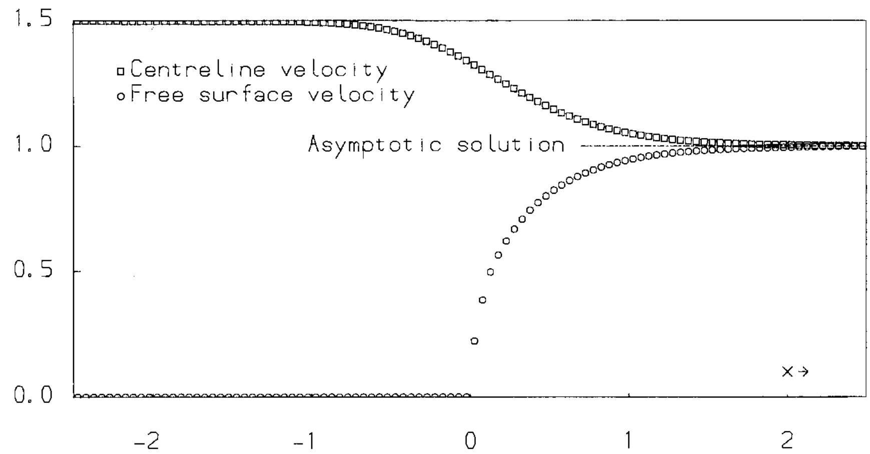

Figure 4. Velocity distributions on the channel centreline and free surface for $T=1000$ and $N=280$.

free surface $y=\eta(x)$ for a surface tension of $T=0.001$. Both velocities are plotted against the distance along the channel centreline and were produced by the 280 node BBIE. Figure 4 shows corresponding results for $T=1000$. The results of Figs. 3 and 4 are graphically indistinguishable from those presented in [7] which were produced by the FE method for the same values of T. Further, the results of Fig. 4 are graphically indistinguishable from Richardson's [10] analytic solution. Figures 3 and 4 illustrate how the downstream velocity attains its asymptotic value as little as 2.5 channel widths away from the initial expansion – a result consistent with the behaviour suggested in [10].

In Table 1 we present the percentage swell values obtained from each BBIE discretisation for each surface tension in the range considered. Also shown are the iterations required in order to satisfy the necessary convergence criteria. Iterations were continued until the swells could be quoted to an accuracy of three significant figures (SF) for the higher surface tensions and four SF for the lower. Had only three SF accuracy been required at each surface tension, the number of iterations required would never had exceeded three for any combination of T and N. Corresponding swells obtained from the FE method [7], where available, are also presented and are seen to be in good agreement

Table 1
A comparison of the percentage swells obtained with the BBIE and the FE from [7]

Non dimensional surface tension T	BIE results						FE results from [7]		
	$N=70$		$N=140$		$N=280$		Mesh I	Mesh II	Mesh III
	% swell	Iters	% swell	Iters	% swell	Iters	% swell	% swell	% swell
1000	0.031	3	0.031	2	0.031	2	–	–	–
100	0.30	3	0.30	2	0.31	2	0.2	0.2	0.2
10	1.92	4	1.99	3	2.02	2	1.8	1.8	1.7
1	10.22	4	10.36	5	10.44	4	10.0	10.1	10.0
0.1	16.47	7	16.56	3	16.54	3	16.5	16.8	17.2
0.01	16.82	6	16.62	4	16.71	3	–	–	–
0.001	17.87	7	17.72	4	17.81	3	17.4	17.7	18.2

with those generated by the BBIE. For completeness we provide the characteristics of the different FE meshes used by Ruschak [7]:

Mesh I $-$ 72 elements, 442 unknowns, $-0.67 \leqslant x \leqslant 3.33, \quad 0 \leqslant y \leqslant 1,$

Mesh II $-$128 elements, 748 unknowns, $-0.75 \leqslant x \leqslant 3.25, \quad 0 \leqslant y \leqslant 1,$

Mesh III $-$230 elements, 1302 unknowns, $-0.75 \leqslant x \leqslant 3.25, \quad 0 \leqslant y \leqslant 1,$

where the regions of solution have been transformed into the coordinates used in the present work. Note that the FE results were obtained in an appreciably smaller solution domain.

4. Conclusions

An iterative BBIE method has been presented which is able to solve viscous free-surface flows for a wide range of surface tensions. Application of the free-surface boundary conditions is very straightforward using the present formulation, even when the boundary location is unknown. The method requires a knowledge of the asymptotic form of the solution for large surface tension and so is applicable to any such problem for which this asymptotic form is obtainable. Unlike the usual space discretisation techniques such as the FD or FE method, the present method requires information on the boundary only and so lends itself ideally to the solution of free surface problems since it does not evaluate data in the interior of the solution domain at each iteration.

It is hoped to develop the method to deal with the flows associated with lubrication technology, for example as in [20,21] as such methods would prove useful in the solution of coating problems [22,23] encountered in polymer processes.

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